Low-energy effective-field theories of Sp(4) spin systems

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We study the classical and quantum phase transitions of Sp(4) spin systems on three-dimensional stacked square and triangular lattices. We present general Ginzburg-Landau field theories for various types of Sp(4) spin orders with different ground-state manifolds such as CP(3), S^7/Z_2, Grassmann manifold $G_{2,5}$, $G_{2,6}$, and so on, based on which the nature of the classical phase transitions are studied, and a global phase diagram is presented. The classical phase transitions close to quantum phase transitions toward spin-liquid states are also discussed based on renormalization group flow. Our results can be directly applied to the simplest Sp(4) and SU(4) Heisenberg models which can be realized using spin-3/2 atoms and alkaline-earth atoms trapped in optical lattice.

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I. INTRODUCTION

For decades condensed-matter physicists have been actively studying the spin systems with large symmetries such as SU(N) and Sp(N),[1–10] mainly motivated by the fact that under large-N generalization the semiclassical spin order with spin symmetry breaking is weakened, and even vanishes completely beyond certain critical N_c. A good example is the SU(N) Heisenberg model on square lattice with fundamental and conjugate fundamental (FCF) representation on two sublattices (FCF Heisenberg model), which for $N>4$ is quantum disordered, and for $N\leq 4$ the ground state spontaneously breaks the SU(N) symmetry, with ground-state manifold (GSM) CP($N-1$).[3,11,12] Very recently it was proposed that, without fine tuning any parameter, the SU(N) spin systems with N as large as 10 can be realized by alkaline-earth atoms trapped in optical lattice,[13] so the large-N spin system is no longer merely theoretical toy. Many previous works showed that for the special value $N=4$, the Sp(4) symmetry can be realized with spin-3/2 fermionic atoms, and when the spin-0 and spin-2 s-wave scattering lengths are equal, the system has an even larger SU(4) symmetry.[14] Motivated by these observations, quantum magnetism based on the spin-3/2 atoms has been actively studied.[14–20]

Although the GSM of the ordered state of SU(N) FCF Heisenberg model on square lattice has been identified as CP($N-1$) long ago, a detailed Ginzburg-Landau (GL) field theory for this ordered state has not been thoroughly studied. A GL theory of this state can answer the following question: suppose the SU(N) Heisenberg model is defined on the three-dimensional (3D) cubic lattice with CP($N-1$) GSM, what is the finite temperature transition between this ordered phase at low temperature and a disordered phase at high temperature? For $N=2$, this question is fairly simple, because CP(1)$=S^2$, the finite temperature transition is not more than one single 3D O(3) transition. For larger-N cases, the question is complicated by the fact that CP($N-1$) manifold does not have a general simple parametrization as CP(1). The standard way to parameterize the CP($N-1$) manifold is to treat it as N component of complex boson coupled with U(1) gauge field, while keeping the SU(N) global symmetry of the action, but this parametrization of CP($N-1$) manifold fails to describe the finite temperature phase transition, which will be discussed in the next section. Therefore we need to write down a GL theory based only on the physical observable order parameters.

In the current work we will focus on the case with $N=4$ and discuss the finite temperature phase transition of system with CP(3) GSM. One sample system which has CP(3) GSM is the Sp(4) Heisenberg model on bipartite lattice with one particle per site

$$H=\sum_{\langle i,j\rangle} J_1\Gamma_i^{ab}\Gamma_j^{ab}-J_2\Gamma_i^a\Gamma_j^a. \tag{1}$$

Γ^a with $a=1,\ldots,5$ are five 4×4 Gamma matrices, and $\Gamma^{ab}=\frac{1}{2i}[\Gamma^a,\Gamma^b]$ are ten generators of Sp(4)$\sim$SO(5) group. Here we choose the following standard convention of Gamma matrices:

$$\Gamma^a=\mu^z\otimes\sigma^a,\quad a=1,2,3,\quad \Gamma^4=\mu^x\otimes\mathbf{1},\quad \Gamma^5=\mu^y\otimes\mathbf{1}. \tag{2}$$

For arbitrary J_1 and J_2 this system has Sp(4) symmetry, while when $J_1=J_2$ this model is equivalent to the SU(4) FCF Heisenberg model.[17] J_1 and J_2 can be tuned with spin-0 and spin-2 s-wave scattering lengths of spin-3/2 cold atoms.[17] Our formalism suggests that for a Sp(4) spin system on the 3D cubic lattice with GSM CP(3), depending on the ratio J_1/J_2 the classical phase diagram can have different scenarios. The most interesting scenario is the region $J_2>J_1$ in model (1), at finite temperature there are two transitions, with one 3D O(5) transition followed by a 3D O(3) transition at lower temperature. On stacked triangular lattice, it was shown that the Sp(4) Heisenberg model (1) has $\sqrt{3}\times\sqrt{3}$ spin order with GSM S^7/Z_2.[21] At finite temperature again there can be two transitions, with one 3D O(5) transition followed by a "coupled" O(3) transition. Besides CP(3) and S^7/Z_2, many other spin symmetry breaking semiclassical states of Sp(4) spins with different GSM can exist, especially for half-filled (two particles per site) system, which will also be discussed in this work.

This paper is organized as follows. In Sec. II, we will study the GL theory of the Neel and $\sqrt{3}\times\sqrt{3}$ phases of the Sp(4) spin system on three-dimensional lattices, and a global phase diagram is presented. Sp(4) spin states with other GSM such as Grassmann manifold $G_{2,5}$, $G_{2,6}$, and SO(5)/SO(3) will also be discussed, with applications to half-filled Sp(4) spin models. Our GL theory can also be used to distinguish different GSMs with the same dimension and similar quotient space representation. In Sec. III we will study the classical phase transitions close to quantum phase transitions between ordered and spin-liquid phases. In Sec. IV we will briefly discuss a more exotic manifold, the "squashed S^7" and its potential to be realized in Sp(4) spin systems.

II. GL THEORIES FOR SP(4) SPIN SYSTEMS

A. Collinear phases

Let us now consider the Sp(4) Heisenberg model (1) on the 3D cubic lattice with one particle per site. On the two-dimensional (2D) square lattice, both analytical and numerical results conclude that at the special point $J_1=J_2$ with enlarged SU(4) symmetry, the ground state of this model has semiclassical order,[3,11,12] with GSM CP(3), which extends into a finite range of the phase diagram tuned by J_2/J_1.[21] The semiclassical order is expected to be stable with the third-direction unfrustrated interlayer coupling. In this Neel phase, both Γ^{ab} and Γ^a are ordered. For instance, we can take the trial single-site state $|\psi\rangle=(1,0,0,0)^t$, and it is trivial to see that it has nonzero Γ^3, Γ^{45}, and Γ^{12}.

As already mentioned in the introduction, the standard way to parameterize the CP($N-1$) manifold is to treat it as N component of complex boson coupled with U(1) gauge field, while keeping the global spin symmetry of the action

$$L=|(\partial_\mu - iA_\mu)z|^2 + r|z|^2 + g(|z|^2)^2 + \cdots . \tag{3}$$

This action is written down based on the fact that

$$\mathrm{CP}(N-1) = S^{2N-1}/\mathrm{U}(1). \tag{4}$$

Here S^{2N-1} represents the GSM of the condensate of N component of complex boson, and U(1) represents the U(1) gauge field A_μ. However, at finite temperature, a simple CP($N-1$) model in Eq. (3) on three spatial dimensions would lead to a wrong transition, because this model describes a transition between the ordered phase and a photon phase. However, finite temperature induces finite density of monopoles of A_μ, which will change the photon propagator at long scale. The disordered phase is generically identical to the "confined phase" with monopole proliferation and no lattice symmetry breaking, i.e., the monopoles without Berry phase. Therefore the action (3) should be supplemented with the "featureless" monopole, which is relevant at least for small N at the critical point $r=0$. For $N=2$, the "trivial" monopole drives the CP(1) model to the O(3) universality class, but for larger N there is no such simple relation. Therefore the CP($N-1$) model plus monopole does not tell us much about the nature of the transition, in general, and we need another convenient way to describe the CP($N-1$) manifold.

Therefore, to describe the GSM and transition we need to introduce a linear sigma model at $4-\epsilon$ dimension with gauge-invariant order parameters, in the form of $z^\dagger_\alpha z_\beta$. There are in total 15 independent bilinears of this form, which can be simply rewritten as the following five-component vector and ten-component adjoint vector:

$$\phi^{ab} = z^\dagger \Gamma_{ab} z, \quad \phi^a = z^\dagger \Gamma_a z$$

$$\sum_{a,b} \phi^{ab}\phi^{ab} \sim \sum_a \phi^a\phi^a \sim (|z|^2)^2. \tag{5}$$

The complex bosonic field z_α are the low-energy Schwinger bosons of Sp(4) spin system. In Ref. 21, it was shown that in the Neel order ϕ^{ab} is the staggered order $(-1)^i\Gamma^{ab}$ while the O(5) vector ϕ^a is the uniform order Γ^a, which can be naturally expected from Eq. (1), when J_1 and J_2 are both positive. However, ϕ^{ab} and ϕ^a are not independent vectors, because the Sp(4) symmetry of the system allows for coupling between these two vectors in the free energy, which can be manifested by the following identities:

$$\sum_{a=1}^{5} \phi^a\phi^a = 2(|z|^2)^2, \quad \sum_a \phi^a N^a = 2(|z|^2)^3,$$

$$N^a = \epsilon_{abcde}\phi^{bc}\phi^{de}. \tag{6}$$

where ϵ_{abcde} is the five-dimensional antisymmetric tensor. Also, the five Γ^a matrices are all constructed by bilinears of the spin-3/2 operators, while Γ^{ab} are constructed by linear and cubics of the spin operators.[14,17] Therefore $\vec{\phi}$ is time reversal even, and identical to the nematic O(5) vector $N^a=\epsilon_{abcde}\phi^{bc}\phi^{de}$ in the ordered state of the CP(3) model with $|z|^2=1$.

Now we can write down a classical GL theory for Sp(4) spin system with CP(3) GSM

$$F = \sum_{ab,\mu} (\nabla_\mu\phi^{ab})^2 + (\nabla_\mu\phi^a)^2 + r_1(\phi^{ab})^2 + r_2(\phi^a)^2 + \gamma\epsilon_{abcde}\phi^a\phi^{bc}\phi^{de} + g\left\{\sum_{ab}(\phi^{ab})^2 + \sum_a(\phi^a)^2\right\}^2 + \cdots . \tag{7}$$

The ellipses include all the other terms allowed by Sp(4) global symmetry. When $r_1=r_2$, this free energy is SO(6) $\sim$ SU(4) invariant, which corresponds to the point $J_1=J_2$, where the model is equivalent to the SU(4) FCF Heisenberg model on the cubic lattice. We can also view the adjoint vector ϕ^{ab} as an O(10) vector which originally should form a GSM S^9, and the cubic term γ makes the ten-component vector ϕ^{ab} align in a six-dimensional submanifold of S^9 where the O(5) vector $\phi^a \sim \epsilon_{abcde}\phi^{bc}\phi^{de}$ is maximized.

A global mean-field phase diagram can be plotted against $r=r_1+r_2$ and $\Delta r=r_1-r_2$, as shown in Fig. 1. The parameter r is tuned by temperature, and Δr is tuned by $\Delta J=J_1-J_2$, which is evident with the observation that $\Delta r=0$ corresponds to the same SU(4) point $\Delta J=0$ and both finite Δr and ΔJ violate the SU(4) symmetry. There are three different regions in the phase diagram. Close to the SU(4) point $\Delta r=0$, the

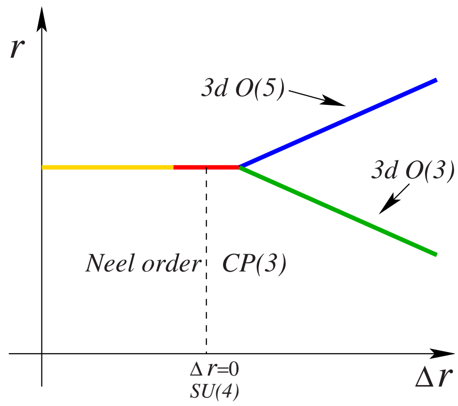

FIG. 1. (Color online) The phase diagram of GL theory, Eq. (7), plotted against $r=r_1+r_2$ and $\Delta r=r_1-r_2$. The red line (the transition around $\Delta r=0$) is a first-order transition, the blue line (upper line with $\Delta r>0$) is a 3D O(5) transition, and the green line (lower line with $\Delta r>0$) is a 3D O(3) transition. The golden line (the transition with $\Delta r<0$) is a second-order transition at the mean-field level, the true nature of the transition can be obtained by a detailed RG calculation for Eq. (8) with $\gamma_2<0$. A similar phase diagram can be applied to Eq. (14) for the stacked triangular lattice, with the green line representing a coupled O(3) transition described by Eq. (16).

cubic term γ drives a first-order transition at the mean-field level, with both $\langle\phi^{ab}\rangle$ and $\langle\phi^{a}\rangle$ jump discontinuously. The first-order transition extends to a finite region in the phase diagram. The second region of the phase diagram has $\Delta r<0$ ($J_1>J_2$), here ϕ^{ab} wants to order before ϕ^a, but due to the γ term in the free energy [Eq. (7)], the order of ϕ^{ab} implies order of ϕ^a. Therefore in this region the phase transition can be safely described by a free energy in terms of only ϕ^{ab}, after integrating out ϕ^a

$$F_2=\sum_{ab,\mu}(\nabla_\mu\phi^{ab})^2+r(\phi^{ab})^2+\gamma_2\sum_a(\epsilon_{abcde}\phi^{bc}\phi^{ed})^2 + g_2\left(\sum_{ab}(\phi^{ab})^2\right)^2+\cdots. \tag{8}$$

Here $\gamma_2<0$ to make sure the ground state wants to maximize ϕ^a. We can treat γ_2 as a perturbation at the 3D O(10) transition, and a coupled renormalization group (RG) flow of γ_2 and g_2 will determine the fate of the transition.

The third region is $\Delta r>0$ ($J_1<J_2$), now ϕ^a tends to order before ϕ^{ab}, and there are, in general, two separate second-order transitions at finite temperature, with ϕ^a orders first. The transition of ϕ^a is a three-dimensional O(5) transition. After the ordering of ϕ^a, the symmetry of the system breaks down to O(4). Let us take the expectation value of $\vec{\phi}$ as $\langle\vec{\phi}\rangle=(\sigma,0,0,0,0)$, the coupling between ϕ^a and ϕ^{ab} in free energy [Eq. (7)] reads

$$\epsilon_{abcde}\phi^a\phi^{bc}\phi^{de}=\sigma(\phi^{23}\phi^{45}-\phi^{24}\phi^{35}+\phi^{25}\phi^{34}). \tag{9}$$

Now one can diagonalize the quadratic part of the Eqs. (7) and (9), the eigenmodes are characterized by the representation of the residual $O(4)\simeq SU(2)\times SU(2)$ symmetry. The residual O(4) symmetry group is generated by six matrices Γ^{ab} with $a,b\neq 1$. The two SU(2) normal subgroups of O(4) are generated by matrices $-\Gamma^{23}+\Gamma^{45}$, $\Gamma^{24}+\Gamma^{35}$, and $-\Gamma^{25}+\Gamma^{34}$ [denoted as subalgebra $su(2)_A$] and $\Gamma^{23}+\Gamma^{45}$, $-\Gamma^{24}+\Gamma^{35}$, and $\Gamma^{25}+\Gamma^{34}$ [denoted as subalgebra $su(2)_B$], respectively. We will decompose the ten-component vector ϕ^{ab} based on the representation of the $su(2)_A$ and $su(2)_B$ algebras, different representations will have different eigenvalues

$$\vec{Q}^i(i=1,\ldots,4)=(\phi^{12},\phi^{13},\phi^{14},\phi^{15}),$$

$$\text{eigenvalue:}r,\quad \text{representation:O(4)vector;}$$

$$\vec{T}_A^i(i=1,2,3)=(-\phi^{23}+\phi^{45},\phi^{24}+\phi^{35},-\phi^{25}+\phi^{34}),$$

$$\text{eigenvalue:}r-\gamma\sigma,\quad \text{representation:}(1,0);$$

$$\vec{T}_B^i(i=1,2,3)=(\phi^{23}+\phi^{45},-\phi^{24}+\phi^{35},\phi^{25}+\phi^{34}),$$

$$\text{eigenvalue:}r+\gamma\sigma,\quad \text{representation:}(0,1). \tag{10}$$

Here $\vec{T}_A$ and $\vec{T}_B$ transform as vectors of $SU(2)_A$ and $SU(2)_B$, respectively. Notice that although $SU(2)_A$ and $SU(2)_B$ are both normal subgroups of the SO(4) after the order of ϕ^a, neither of them can be normal subgroup of the original SO(5) group, because SO(5) group is a simple group while SO(4) is a semisimple group.

If $\gamma\sigma>0$, $\vec{T}_A$ has the lowest eigenvalue, so the O(3) vector $\vec{T}_A$ will order after ϕ^a. The main question is which universality this transition belongs to. Since $\vec{Q}$ and $\vec{T}_B$ are massive and only have short-range correlation at the transition of $\vec{T}_A$, integrating out them will not induce any critical behavior for $\vec{T}_A$, and hence the Goldstone mode of ϕ^a after its ordering is the biggest concern. The Goldstone mode $(0,\pi_1,\pi_2,\pi_3,\pi_4)$ forms an O(4) vector, and the Goldstone theorem guarantees its gaplessness. The simplest coupling one can write down with these constraints is

$$F'\sim(\vec{T}_A)^2(\nabla_\mu\vec{\pi})^2. \tag{11}$$

This term only generates irrelevant perturbations at the O(3) transition of $\vec{T}_A$ after integrating out $\vec{\pi}$. Notice that couplings such as $(\vec{T}_A)^2(\vec{\pi})^2$ though preserves the global O(4) symmetry, violates the Goldstone theorem after integrating out $\vec{T}_A$, as a mass gap $\sim\langle\vec{T}_A^2\rangle$ is induced for $\vec{\pi}$. Therefore now we can safely conclude that the phase transition of $\vec{T}_A$ is a 3D O(3) transition. Notice that vectors $\vec{T}_B$ and $\vec{Q}$ no longer have to order at lower temperature, because of the repulsion from ordered $\vec{T}_A$, due to the quartic terms in Eq. (7).

After the ordering of $\vec{T}_A$, the symmetry of the system is broken down to $SO(2)\times SO(3)$. The first SO(2) corresponds to the residual symmetry of $SU(2)_A$ after the order of $\vec{T}_A$, and the second SO(3) corresponds to the $SU(2)_B$ associated with $\vec{T}_B$, therefore CP(3) manifold can also be written as quotient space $SO(5)/[SO(2)\times SO(3)]$. However, we should be careful about this formula, because there are two different types of so(3) or su(2) subalgebras of so(5). Besides the subalgebras $su(2)_A$ and $su(2)_B$ we used earlier, there is another SU(2) subgroup which is the diagonal subgroup of

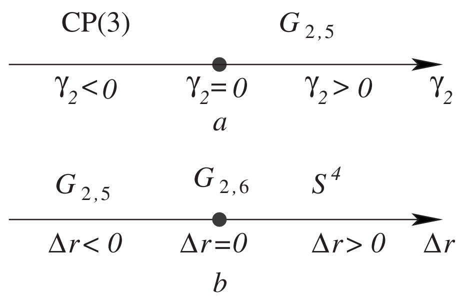

FIG. 2. The schematic ground-state manifold phase of (a) [Eq. (8)] and (b) [Eq. (12)], with $\Delta r = r_1 - r_2$.

$SU(2)_A \times SU(2)_B$, we denote this subgroup as $SU(2)_V$, which is no longer a normal subgroup of O(4). The elements in algebra $su(2)_V$ are the linear combination of the corresponding elements in $su(2)_A$ and $su(2)_B$: $J_i^V = J_i^A + J_i^B$.

For instance, in the half-filled (two particles per site) spin-3/2 cold atoms, one can naturally obtain an ordered state with $\langle(-1)^i\Gamma^{ab}\rangle \neq 0$ but with no order of Γ^a,[19,20] which means that for this case action (8) is still applicable, while the sign of γ_2 is positive, i.e., it corresponds to a different anisotropy of the S^9 manifold formed by the adjoint vector ϕ^{ab}, which minimizes the vector $\phi^a \sim \epsilon_{abcde}\phi^{bc}\phi^{de}$ [in contrast to CP(3)] (Fig. 2). In this case the GSM can still be written as $SO(5)/[SO(2)\times SO(3)]$, but here SO(3) is $SU(2)_V$. For instance, if $\langle(-1)^i\Gamma^{12}\rangle \neq 0$, the $SU(2)_V$ is generated by Γ^{34}, Γ^{35}, and Γ^{45}. This GSM $SO(5)/[SO(2)\times SO(3)]$ with $SO(3)\sim SU(2)_V$ is called Grassmann manifold $G_{2,5}$, which is mathematically defined as the set of two-dimensional planes in five-dimensional vector space.[22]

The mean-field phase diagram for the half-filled Sp(4) system tuned by the spin-0 and spin-2 s-wave scattering lengths is studied in Refs. 14, 19, and 20. Besides the phase with $\langle(-1)^i\Gamma^{ab}\rangle \neq 0$ discussed in the previous paragraph, there is another typical phase with $\langle(-1)^i\Gamma^{a}\rangle \neq 0$ and GSM $SO(5)/SO(4) = S^4$. These two phases are separated from each other by the SU(4) point with equal spin-0 and spin-2 scattering lengths, where due to the enlarged symmetry, the two different orders should have equal energy.[14] Suppose $\langle(-1)^i\Gamma^{12}\rangle$ is nonzero at this SU(4) point, now the residual symmetry of this order is generated by Γ^{12}, Γ^{34}, Γ^{45}, Γ^{35}, Γ^3, Γ^4, and Γ^5, which form subgroup $SO(2)\times SO(4)$ of the $SO(6)\sim SU(4)$ symmetry group. More detailed analysis would show that now the GSM is the Grassmann manifold $SO(6)/[SO(2)\times SO(4)] = G_{2,6}$ (Fig. 2), which is defined as the set of two-dimensional planes in six-dimensional vector space.

One can write down a GL field theory for the half-filled Sp(4) spin system as follows:

$$F_{hf} = \sum_{ab,\mu} (\nabla_\mu \phi^{ab})^2 + (\nabla_\mu \phi^a)^2 + r_1(\phi^{ab})^2 + r_2(\phi^a)^2 + g\left\{\sum_{ab}(\phi^{ab})^2 + \sum_a (\phi^a)^2\right\}^2 + \sum_a \gamma_2(\epsilon_{abcde}\phi^{bc}\phi^{de})^2 + \cdots. \tag{12}$$

$r_1 = r_2$ corresponds to the SU(4) point, and $r_2 < r_1$ ($r_2 > r_1$) corresponds to the case with $\langle(-1)^i\Gamma^a\rangle \neq 0$ ($\langle(-1)^i\Gamma^{ab}\rangle \neq 0$). Notice that the cubic term $\gamma\epsilon_{abcde}\phi^a\phi^{bc}\phi^{de}$ is not allowed here because ϕ^a and ϕ^{ab} both represent staggered orders, so this cubic term would switch sign under lattice translation. The ellipses in Eq. (12) includes other terms allowed by symmetry, for instance, $\sum_{ab}(\epsilon_{abcde}\phi^c\phi^{de})^2$.

In 2+1-dimensional space, another possible ground state around the SU(4) point of the half-filled system is the algebraic spin liquid, which has been actively studied analytically[7,8,23–26] and has gained numerical supports.[27] However, the fate of the SU(4) point at three dimensions is unclear, so in this work we tentatively assume it still has magnetic order which bridges the orders on two sides of the phase diagram in Fig. 2(b), and the transition between the two different spin order patterns at zero temperature should be first order.

B. Noncollinear phases

Now let us move on to the GL theory for Sp(4) spin system with noncollinear spin orders. It was shown[21] that the GSM of the ordered phase of Sp(4) system on the triangular lattice is S^7/Z^2 with $\sqrt{3}\times\sqrt{3}$ order of Γ^{ab} and collinear and uniform order of Γ^a. By tuning J_2/J_1 there is a transition between the ordered phase and a deconfined Z_2 spin liquid which belongs to the 3D O(8) universality class. Now let us consider the Sp(4) Heisenberg model on the stacked triangular lattice, and study the GL theory in terms of physical order parameters. This ordered state is characterized by the $\sqrt{3}\times\sqrt{3}$ order of $\phi_1^{ab} + i\phi_2^{ab} = z^t\Gamma^{ab}z$, and a uniform order of $\phi^a = z^\dagger\Gamma^a z$. z_α is the Sp(4) bosonic spinon expanded at the minima of the spinon band structure, which are located at the corners of the hexagonal Brillouin zone $\vec{Q} = (\pm 4\pi/3, 0)$. The two ten-component Sp(4) adjoint vectors ϕ_1^{ab} and ϕ_2^{ab} are "perpendicular" to each other: $\sum_{a,b}\phi_1^{ab}\phi_2^{ab} = 0$. In the ordered state, The vectors ϕ_1^{ab}, ϕ_2^{ab}, and ϕ^a satisfy the following relations:

$$\epsilon_{abcde}\phi_1^{bc}\phi_1^{de} = \epsilon_{abcde}\phi_2^{bc}\phi_2^{de} \sim |z|^2\phi^a. \tag{13}$$

Therefore the GL theory reads

$$F = \sum_{i=1}^{2}\sum_{a,b} (\nabla_\mu \phi_i^{ab})^2 + r_1(\phi_i^{ab})^2 + (\nabla_\mu\phi^a)^2 + r_2(\phi^a)^2 + \sum_i \gamma\epsilon_{abcde}\phi^a\phi_i^{bc}\phi_i^{de} + g_3\left[\sum_{ab,i}(\phi_i^{ab})^2\right]^2 + g_4\left\{\left(\sum_{ab}\phi_1^{ab}\phi_2^{ab}\right)^2 - \left[\sum_{ab}(\phi_1^{ab})^2\right]\left[\sum_{cd}(\phi_2^{cd})^2\right]\right\}. \tag{14}$$

The last term in Eq. (14) with $g_4 > 0$ guarantees the "orthogonality" between ϕ_1^{ab} and ϕ_2^{ab} in the ordered phase. Besides the apparent Sp(4) symmetry, this free energy, Eq. (14), within the forth order has an extra O(2) symmetry for rotation between ϕ_1^{ab} and ϕ_2^{ab}, which corresponds to the translation symmetry of the system

$$T_x : \phi_1^{ab} + i\phi_2^{ab} \rightarrow (\phi_1^{ab} + i\phi_2^{ab})\exp(i2\pi/3). \quad (15)$$

For the commensurate $\sqrt{3}\times\sqrt{3}$ order, this O(2) symmetry will be broken by the sixth order terms of this free energy; if the noncollinear state is incommensurate, the O(2) symmetry will be preserved by any higher order of the GL theory.

In the GL theory, Eq. (14), depending on $\Delta r = r_2 - r_1$, the order of ϕ^a is allowed to occur before the order of ϕ_i^{ab}, and the transition of ϕ^a again belongs to the O(5) universality class. After the order of ϕ^a, the quadratic part of the free energy, Eq. (14), can be diagonalized, and O(3) vectors $\vec{T}_{A,1}$ and $\vec{T}_{A,2}$ would order after ϕ^a. The last term in Eq. (14) would induce a term $(\vec{T}_{A,1}\cdot\vec{T}_{A,2})^2$ at this transition, therefore the field theory for the second transition is described by the following coupled O(3) free energy:

$$F = \sum_{i=1}^{2} (\nabla_\mu \vec{n}_i)^2 + r(\vec{n}_i)^2 + v\{(\vec{n}_1)^2 + (\vec{n}_2)^2\}^2 + u\{(\vec{n}_1\cdot\vec{n}_2)^2 - (\vec{n}_1)^2(\vec{n}_2)^2\} + \cdots \quad (16)$$

with $\vec{n}_i = \vec{T}_{A,i}$. Again the Goldstone mode of ϕ^a only induces irrelevant perturbation. This coupled O(3) model defined in Ref. 28 with symmetry $O(2)\times O(3)$ has attracted enormous analytical and numerical work, recent results suggest the existence of a new universality class of the coupled O(3) model.[29] When n_1 and n_2 are ordered, the whole SO(3) symmetry associated with $\vec{T}_{A,i}$ is broken, and the residual symmetry of the condensate of $\vec{n}_i$ is SO(3), which is the SO(3) symmetry associated with $\vec{T}_{B,i}$, i.e., $SU(2)_B$.

Again the nature of the GSM depends on which type of SO(3) the residual symmetry is. For half-filled spin-3/2 cold atoms on the triangular lattice, one can engineer a state without order of Γ^{ab}, but with $\sqrt{3}\times\sqrt{3}$ order of nematic order parameter Γ^a

$$\langle \Gamma^a(\vec{r})\rangle \sim n_1^a \cos(\vec{Q}\cdot\vec{r}) + i n_2^a \sin(\vec{Q}\cdot\vec{r}),$$

$$\sum_{a=1}^{5} n_1^a n_2^a = 0. \quad (17)$$

This spiral nematic order parameter has residual symmetry SO(3), however, this is the $SU(2)_V$ subgroup discussed previously. For instance, if $\vec{n}_1 = (1,0,0,0,0)$ and $\vec{n}_2 = (0,1,0,0,0)$ then $SU(2)_V$ is generated by Γ_{34}, Γ_{45}, and Γ_{35}. Therefore the GSM of this order can be written as quotient space SO(5)/SO(3), but not equivalent to S^7/Z_2. The GL theory describing this nematic $\sqrt{3}\times\sqrt{3}$ order is a coupled O(5) sigma model, which is analogous to Eq. (16).

Another state worth mentioning briefly is the superconductor state of the Sp(4) fermions, and we will only focus on the s-wave pairing here. The s-wave pairing of two Sp(4) particles can be either Sp(4) singlet or quintet. And the quintet state which is characterized by a complex O(5) vector $\vec{d} = \vec{d}_1 + i\vec{d}_2$ can have two types of GSM, depending on the microscopic parameters of the system. The first type of pairing has $\vec{d}_1$ parallel with $\vec{d}_2$, then the GSM is $[S^4\times S^1]/Z_2$.[30] The second type of pairing has $\vec{d}_1\cdot\vec{d}_2 = 0$, then the GSM is again characterized by two real orthogonal O(5) vectors, and hence GSM=SO(5)/SO(3), equivalent to the nematic $\sqrt{3}\times\sqrt{3}$ state discussed in the previous paragraph. In experimental system with spin-3/2 cold atoms, the direct calculation with s-wave scattering suggests that the former state (dubbed polar state) is likely favored.[17]

III. CLOSE TO QUANTUM PHASE TRANSITIONS

In this section we will study the phase transitions obtained in the previous section in the region close to a quantum phase transition. For two-dimensional square lattice, it was proposed in Ref. 21 that by tuning J_2/J_1 in Eq. (1), there is a deconfined quantum phase transition between Neel order and a gapped plaquette order which belongs to the 3D CP(3) universality class. If now we turn on a weak spin interaction between square lattice layers, the deconfined quantum phase transition is expected to expand into a stable spin-liquid phase with gapless photon excitation, while the Neel order and plaquette order are unaltered by the weak z-direction tunnelling.

The quantum phase transition between Neel and photon phase is described by the $3+1d$ CP(3) model

$$L = \sum_{a=1}^{4} |(\partial_\mu - iA_\mu)z_a|^2 + r|z_a|^2 + g(|z_a|^2)^2 + \frac{1}{16e^2}F_{\mu\nu}^2 + \cdots. \quad (18)$$

Based on naive power counting this $3+1d$ transition is a mean-field theory with marginally relevant/irrelevant perturbations. To determine the universality class of this transition, we need to calculate the RG equation for g and e^2 in Eq. (18) in detail. At the transition with $r=0$, the coupled RG equation up to one loop for g and e^2 reads

$$\frac{dg}{d\ln l} = -\frac{2}{\pi^2}g^2 - \frac{3}{8\pi^2}e^4 + \frac{3}{4\pi^2}e^2 g,$$

$$\frac{de^2}{d\ln l} = -\frac{1}{6\pi^2}e^4. \quad (19)$$

The RG equation for the Higgs model with $N=1$ was calculated in Ref. 31, the structure of the RG equation obtained therein is quite similar to Eq. (18). Taking this RG equation, one can see that the electric charge e^2 is always renormalized small. If one starts with a positive value of g, g will be first renormalized to smaller values marginally, and then switch sign due to its coupling with e^2, and finally becomes nonperturbative, and no fixed point is found with arbitrary choices of initial values of g and e^2. So eventually this transition is probably weak first order. The solution of RG equations (18) is plotted in Fig. 3 for the trial initial value $g_0 = e_0^2 = 1/5$. One can see that g becomes nonperturbative much slower than an ordinary marginally relevant operator, because the ordinary marginally relevant operator will still monotonically increase under RG flow. In our current case g remains perturbative and decreases for a very large energy scale, so for sufficiently small initial values of g and e^2, at physically relevant energy scale, we can treat this transition a mean-field transition of spinon z_α.

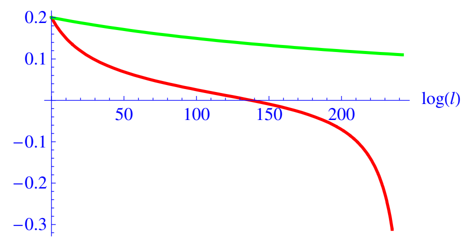

FIG. 3. (Color online) The RG flow for e^2 (green upper line) and g (red lower line) in Eq. (18) with trial initial value $g_0 = e_0^2 = 1/5$.

Without monopoles, the finite temperature transition will be described by the 3D CP(3) model in Eq. (3). If temperature is turned on, finite density of monopoles will be generated. Close to the quantum transition, since the critical temperature of the Neel order is very low, the monopoles roughly have small fugacity $y_m \sim \exp(-E_g/T)$, and E_g represents the short-distance energy gap of monopole. Therefore very close to the quantum phase transition with small T_c, there is a very narrow "monopole dominated" region around the classical phase transition where the universal physics significantly deviates from the CP(3) model. Inside the monopole dominated region the GL field theory in Eq. (7) becomes applicable, with $r = r_1 + r_2$ tuned by temperature. Out of this monopole dominated region, the scaling behavior of the 3D CP(3) model becomes more applicable, assuming the noncompact CP(3) model has a second-order transition. The size of the monopole dominated region can be estimated from the fugacity of the monopoles. If the scaling dimension of the monopole operator at the CP(3) fixed point is Δ_m, the size of the monopole dominated range is estimated as $\Delta T/T_c \sim y_m^{1/(3-\Delta_m)\nu}$, ν is the standard exponent of 3D CP(3) transition defined as $\xi \sim r^{-\nu}$. The phase diagram is shown in Fig. 4.

The situation is quite different for the stacked triangular lattice. In Ref. 21 we showed that on 2D triangular lattice, by tuning J_2/J_1 there is a 3D O(8) transition between the $\sqrt{3}\times\sqrt{3}$ order and the Z_2 spin-liquid state, despite the fact that the microscopic system only has $Sp(4) \sim SO(5) \subset SO(8)$ symmetry. For a stacked triangular lattice with weak interlayer coupling, both the $\sqrt{3}\times\sqrt{3}$ order and the Z_2 spin liquid will survive, but the quantum phase transition is described by the mean-field theory of z_α, because the Z_2 spin liquid does not introduce any critical correlation for z_α. Notice that at this mean-field transition the magnetic order parameters ϕ^{ab} will have anomalous dimension one, because it is a bilinear of z_α. For 3D space, the Z_2 spin liquid can survive and extend into a finite region in the phase diagram at finite temperature, therefore close to the quantum transition, after the thermal fluctuation destroys the magnetic order, the system does not enter the high-temperature featureless phase immediately, instead it enters the finite temperature Z_2 spin-liquid phase, and the classical transition of the spin order will simply belong to the 3D O(8) universality class. At even higher temperature, there is a phase transition separating the classical Z_2 spin-liquid state and high-temperature disordered phase, which physically corresponds to the proliferation of the "vison loop." This transition belongs to the 3D Ising universality class.

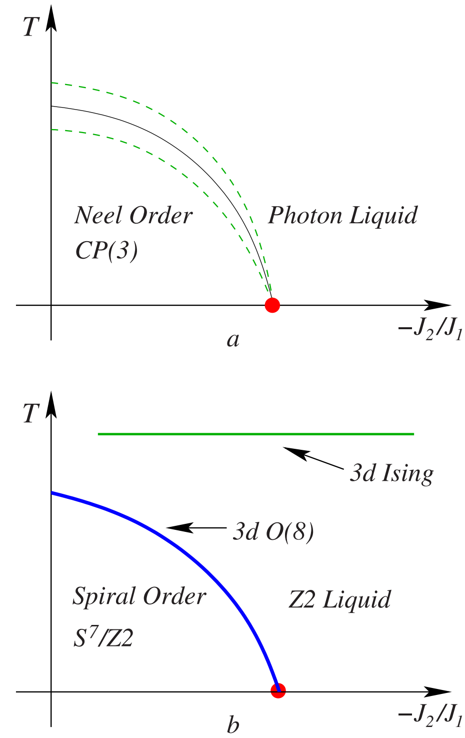

FIG. 4. (Color online) The phase diagram close to the quantum phase transitions in (a) stacked square and (b) triangular lattices. The region between the dashed lines in figure a is the monopole dominated region which should be described by GL theory [Eq. (7)]. The blue lower curve in figure b is a 3D O(8) transition, and the green upper line is a 3D Ising transition which separates a low-temperature classical Z_2 spin liquid from a high-temperature featureless disordered phase.

IV. SUMMARY AND OUTLOOK

In this work we used the Ginzburg-Landau field theory to describe and classify Sp(4) spin orders with different ground-state manifolds, and studied the nature of classical phase transitions between these spin order and disordered phases. Our results can be applied to Sp(4) spin models such as the $J_1 - J_2$ Heisenberg model in Eq. (1). The GL theory can be generalized for large N spin systems with GSM CP($N-1$), for instance, the cubic term in Eq. (7) is always allowed by spin symmetry for large N, although other discrete symmetries have to be checked carefully.

The monopole of the gauge field A_μ will create and annihilate the quantized flux of A_μ, which equals to the soliton number of the GSM CP($N-1$), and the existence of soliton of system with GSM CP($N-1$) is due to the fact that $\pi_2[\mathrm{CP}(N-1)] = Z$ for general N.[3] If we start with a nonlinear sigma model for CP($N-1$) manifold at $2+\epsilon$ dimension, though the spin wave excitations can be quite nicely described, the expansion of ϵ at the phase transition will not take into account of the effect of monopoles. Therefore the $2+\epsilon$ expansion with extrapolation $\epsilon \to 1$ is probably equivalent to the CP($N-1$) model in Eq. (3). However, if we start with a linear sigma model at $4-\epsilon$ dimension, the $4-\epsilon$ expansion will contain the information of monopoles, and the limit $\epsilon \to 1$ is likely converging to the true situation at three dimensions. Since phase transition is what we are most interested in, in this work we were focusing on the linear sigma model in $4-\epsilon$ dimension.

Another manifold which potentially can be realized by Sp(4) spin system is the "squashed S^7." The squashed S^7 has been studied for over two decades in high-energy theory, as one of the solutions of the 11-dimensional supergravity field equation is $AdS_4 \times S^7_{\text{squash}}$.[32] The squashed S^7 is a seven-dimensional manifold with the same topology as S^7, but different metric and isometry groups. The ordinary S^7 has isometry group SO(8), and the squashed S^7 has isometry group $SO(5)\times SO(3) \subset SO(8)$, and the SO(5) and SO(3) commute with each other. Written as a quotient space, the squashed S^7 can be expressed as[33]

$$S^7_{\text{squash}} = [SO(5) \times SO(3)_C]/[SO(3)_A \times SO(3)_D]. \quad (20)$$

Here $SO(3)_A$ is a normal subgroup of one of the SO(4) subgroup of the SO(5) group in the numerator, and the other normal SO(3) subgroup of this SO(4) is denoted as $SO(3)_B$, i.e., $SO(3)_A \times SO(3)_B \sim SO(4)$. $SO(3)_D$ is the diagonal subgroup of $SO(3)_B \times SO(3)_C$, i.e., $J_i^D = J_i^B + J_i^C$, $i=1,2,3$. To realize the squashed S^7 GSM, we should start with a system with global symmetry $SO(5)\times SO(3)$. For instance, by tuning the two s-wave scattering lengths, the half-filled Hubbard model of the Sp(4) fermions can have an extra SU(2) symmetry besides the apparent Sp(4) flavor symmetry.[14] Also a Sp(4) spin-liquid theory with fermionic spinons with momentum space valley degeneracy can have an extra SU(2) symmetry contributed by the valley degeneracy. So both cases might be a good starting point for realizing the squashed S^7 manifold. We will leave the discussion of squashed S^7 to future study.

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